이학석사 학위논문

Remarks on the Schrödinger-Lohe model

(슈뢰딩거-로헤 모델에 대한 고찰)

2018년 8월

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이 논문을 이학석사 학위논문으로 제출함

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Remarks on the Schrödinger-Lohe model

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Abstract

In this thesis, we study the asymptotic behavior of the coupled Schrödinger-Lohe system under the same one-body potential with a special communication network between particles. First, we review the previous results on the classical all-to-all Schrödinger-Lohe system and the relation between the Schrödinger-Lohe system and Lotka-Volterra system. Then, we present a large-time behavior of the system by introducing the pairwise correlation function. Furthermore, we provide the existence of equilibria for the finite-dimensional system under the general network framework. Finally, we provide several numerical simulation results supporting our analytical result.

Key words: Schrödinger-Lohe model, Lotka-Volterra model, Kuramoto model, dynamical system, quantum synchronization
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Contents

Chapter 1

Introduction

Collective dynamics and synchronization can be easily found in our near nature. Swimming of school of fishes, swarming of bacteria, clapping of the hands in a concert hall or firing of interacting neurons in our brain are examples of such behaviors. It is interesting to construct mathematical models describing such phenomena in nature and analysis to derive a meaningful conclusion. In this context, there have been numerous models to deal with collective phenomena mathematically. The research on the synchronization of phase oscillator was first presented by Winfree and Kuramoto in [15, 16, 17]. After their research, there have been numerous study on the synchronization model. Among them, our main interest is on the Schrödinger-Lohe (S-L) model proposed by M. Lohe [12]. In this thesis, we study the collective dynamics of S-L oscillators under cooperative-competitive interaction. So far, S-L oscillator under all-to-all network or restricted cooperative-competitive interaction has been studied in [2, 3, 5, 6, 13, 14]. In this thesis, we will further generalize the interaction between particles. So, we consider a network (N, A) consisting of N nodes and communication (or interaction) matrix $A = (a_{ij})$ where the element a_{ij} describes the communication weight from the j-th particle to the i-th particle. In here, the value (or amount, weight) of a_{ij} represent the relationship between the nodes: if $a_{ij} > 0$, then i and j nodes is cooperative (or attractive), conversely if $a_{ij} < 0$, then i and j nodes is competitive (or repulsive).

Suppose that i-th S-L oscillator is denoted by $\psi_i = \psi_i(x, t)$ which is the quantum wave function on a periodic spatial domain $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ and these ψ_i is governed by the initial problem to the S-L model: for $i = 1, \cdots, N$,

$$
\begin{aligned}
&\mathrm{i}\partial_t \psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N} a_{ik}(\psi_k - \langle \psi_i, \psi_k\rangle \psi_i), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+, \\
&\psi_i(x,0) = \psi_i^0(x), \quad x \in \mathbb{T}^d, \quad \|\psi_i^0\|_{L^2} = 1,
\end{aligned}
\tag{1.0.1}
$$

where $V_i = V_i(x)$ and κ are real valued one-body potential and coupling strength respectively, and $\langle\cdot,\cdot\rangle$ is an L^2-inner product defined by $\langle \psi_i, \psi_k\rangle = \int_{\mathbb{T}^d} \psi_i \bar{\psi}_k dx$ on $\mathbb{T}^d$. However, we will focus on simpler case. We assume that the dynamics of the wave function of the S-L oscillators over the network whose connection topology is a priori determined by the capacity of the sender, i.e.

$$a_{ik} = \xi_k.$$

Then, the S-L model (1.0.1) can be written as

$$\mathrm{i}\partial_t \psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N} \xi_k(\psi_k - \langle \psi_i, \psi_k\rangle \psi_i). \tag{1.0.2}$$

Moreover, we consider the finite-dimensional reduced equation of (1.0.2). To fix an idea, we consider the pairwise correlation function of two wave functions ψ_i and ψ_j defined as

$$h_{ij} := \langle \psi_i, \psi_j \rangle,$$

which has been introduced in previous works [2, 13]. And we assume that one-body potential satisfy

$$V_i(x) = V(x) + \omega_i, \quad \omega_i \in \mathbb{R}.$$

Then, the dynamics of this correlation function can be deduced from the S-L model (1.0.2) as follows:

$$\dot{h}_{ij} = -\mathrm{i}\omega_{ij} h_{ij} + \frac{\kappa}{N}\sum_{k=1}^{N} \xi_k (h_{ik} + h_{kj})(1 - h_{ij}), \quad t > 0,\ 1 \le i, j \le N,$$

where $\omega_{ij} := \omega_i - \omega_j$. In this thesis, we only consider the simplest case when $\omega_{ij} = 0$. Therefore, the following coupled S-L system is our main object ordinary differential equation:

$$\dot{h_{ij}} = \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k h_{ik} + \sum_{k\neq j}^{N} \xi_k h_{kj}\right)(1 - h_{ij}). \tag{1.0.3}$$

Note that the equation (1.0.3) can be interpreted as the variation of standard Lotka-Volterra (L-V) equation which is introduced by A. Lotka [18, 19] and V.Volterra [19, 20, 21].

It is well-known that there exists a stable equilibrium to equation (1.0.3). Therefore, we are interested in the following question.

Question. Can we determine how many stable equilibriums exist? If so, can we find the exact stable equilibrium?

The remaining thesis is organized as follows. In Chapter 2, we briefly review the Lotka-Volterra system and Schrödinger-Lohe system. We also provide the review on the wave function synchronization in this Chapter. Chapter 3 contains formal reduction of Schödinger-Lohe model into finite-dimensional system (1.0.3) by considering their correlations. In Chapter 4, we investigate equilibria in finite-dimensional system (1.0.3) and answer the question arisen above. The numerical simulations of the finite-dimensional system supporting our main result stated in Chapter 4 are presented in Chapter 5. Finally, Chapter 6 is devoted to a brief summary and conclusion of the thesis.

Notation: Throughout the thesis, for measurable function f defined on $\mathbb{T}^d$, we set

$$\|f\|_2 := \|f\|_{L_2}, \quad \|f\|_\infty := \|f\|_{L_\infty}.$$

Chapter 2

Preliminaries

In this Chapter, we briefly provide the theoretical minimum properties of the L-V system and S-L system. Furthermore, we also review about wave function synchronization.

2.1 The Lotka-Volterra system

In this section, we discuss the steady state of the L-V system. Like rabbits and foxes, the L-V system represents the relationship between the predator and the prey in the ecological environment as the change in the number of the population over time. When $N = 2$, the predator-prey system for prey x and predator y can be described as following ordinary differential equations:

$$\frac{dx}{dt} = ax - bxy = x(a - by),$$
$$\frac{dy}{dt} = -cy + dxy = y(-c + dx),$$

where a, b, c and d are all positive constants, which describes the interaction between prey and predator: a and c are growth and death rate of the prey and predator respectively, and b and d are interaction force between predator and prey. This model adopts the exponential growth model for prey when the absence of predator and it also model the interaction between prey and predator as xy term, which is the simplest choice of interaction form. So that, when a predator is absent, i.e., $y = 0$ we can see that number of prey is exponentially growing at the rate a. Conversely, when prey is absent, i.e., $x = 0$ we can see that number of predator is exponentially decaying at the rate c.

Now, we consider the generalized L-V system. When there are N prey and predator each, the system can be written as follows: for $i = 1, \cdots, N$,

$$\begin{aligned} \frac{d}{dt}x_i &= x_i \left[a_i - \sum_{j=1}^{N} b_{ij} y_j \right], \\ \frac{d}{dt}y_i &= y_i \left[-c_i + \sum_{j=1}^{N} d_{ij} x_j \right], \end{aligned} \tag{2.1.1}$$

where the a_i, b_i, c_i, d_i is all positive constant. And as above, a_i and c_i are growth and death rate of the prey and predator respectively, and b_{ij} and d_{ij} are interaction force between predator and prey. Here, we can show the steady state of (2.1.1) [22]. For trivial steady state $x_i = y_i = 0$, the interactive matrix as follow:

$$M = \left(\begin{array}{ccc|ccc} a_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & a_N & & & \\ \hline & & & -c_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -c_N \end{array} \right).$$

Then eigenvalues of M are a_i and $-c_i$ for $i = 1, \cdots, N$. Since a_i and c_i are positive, M has both positive and negative eigenvalue. So trivial steady state of L-V system is unstable.

Now we focus on the nontrivial steady state. Let $\mathbf{X}$ and $\mathbf{Y}$ be a $N \times 1$ nontrivial steady state vector, then they satisfy

$$a_i - \sum_{j=1}^{N} b_{ij}\mathbf{Y} = 0, \quad -c_i + \sum_{j=1}^{N} d_{ij}\mathbf{X} = 0.$$

Define $N \times N$ communication matrix as $\mathbf{B} := (b_{ij})$, $\mathbf{D} := (d_{ij})$ and $N \times 1$ growth and death rate vector as $\mathbf{a} := (a_j)$, $\mathbf{c} := (c_j)$ for $i, j = 1, \cdots, N$, then

(2.1.1) can be written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}^T \cdot [\mathbf{a} - \mathbf{B}\mathbf{y}], \quad \frac{d\mathbf{y}}{dt} = \mathbf{y}^T \cdot [\mathbf{c} - \mathbf{D}\mathbf{x}],$$

where $\mathbf{x} := (x_i)$ and $\mathbf{y} := (y_i)$. So if we set λ_i be a eigenvalue of M, then it satisfies

$$\sum_{k=1}^{2N} \lambda_k = tr(A) = 0.$$

So that, we can talk about two cases according to $\mathrm{Re}(\lambda_i)$: $\mathrm{Re}(\lambda_i) = 0$ or $\mathrm{Re}(\lambda_i) \neq 0$. If $\mathrm{Re}(\lambda_i) = 0$ for all i, then M only has purely imaginary eigenvalue. So the nontrivial steady state $(\mathbf{X}, \mathbf{Y})$ is neutrally stable. But if there exist at least one i such that $\mathrm{Re}(\lambda_i) \neq 0$, then it implies that there exist at least one i such that $Re(\lambda_i) > 0$. So the nontrivial steady state $(\mathbf{X}, \mathbf{Y})$ is unstable.

2.2 The Schrödinger-Lohe system

In this section, we present a basic property of the S-L system and relation between the S-L system and the Kuramoto system. The wave function of the i-th subsystem at the i-th node $\psi_i = \psi_i(x,t)$ satisfy

$$\mathrm{i}\partial_t \psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N} \xi_k(\psi_k - \langle \psi_i, \psi_k \rangle \psi_i), \quad i = 1, \cdots, N.$$

Then we can show a priori L^2-conservation law.

Lemma 2.2.1. *Let ψ_i be a solution to* (1.0.2) *with $\|\psi_i^0\|_2 = 0$. Then, we have*

$$\|\psi_i(t)\| = 1, \quad t > 0, \quad 1 \leq i \leq N.$$

Proof. First, we can rewritten the system (1.0.2) as

$$\partial_t \psi_i = \mathrm{i}\Delta\psi_i - \mathrm{i}V_i\psi_i + \frac{\kappa}{N}\sum_{k=1}^{N} \xi_k(\psi_k - \langle \psi_i, \psi_k \rangle \psi_i).$$

From this, we take the L^2-inner product with ψ_i. Then we have

$$\langle \psi_i, \partial_t \psi_i \rangle = -\mathrm{i}\langle \psi_i, \Delta\psi_i \rangle + \mathrm{i}\langle \psi_i, V_i\psi_i \rangle + \frac{\kappa}{N}\sum_{k=1}^{N} \xi_k(\langle \psi_i, \psi_k \rangle - \langle \psi_i, \psi_k \rangle \|\psi_i\|_2^2). \tag{2.2.2}$$

We take the complex conjugate of (2.2.2) and use the fact that $\overline{\langle f, g\rangle} = \langle g, f\rangle$, then we get

$$\langle \partial_t \psi_i, \psi_i \rangle = \mathrm{i}\langle \Delta\psi_i, \psi_i \rangle - \mathrm{i}\langle V_i\psi_i, \psi_i \rangle + \frac{\kappa}{N}\sum_{k=1}^{N} \xi_k(\langle \psi_k, \psi_i \rangle - \langle \psi_k, \psi_i \rangle \|\psi_i\|_2^2). \tag{2.2.3}$$

By integration by parts twice using the periodic boundary conditions, we obtain

$$\langle \psi_i, \Delta\psi_i \rangle = \langle \Delta\psi_i, \psi_i \rangle, \quad \langle \psi_i, V_i\psi_i \rangle = \langle V_i\psi_i, \psi_i \rangle. \tag{2.2.4}$$

From (2.2.4), add (2.2.2) and (2.2.3) to find

$$\begin{aligned} \frac{d}{dt}(\|\psi_i\|_2^2 - 1) &= \frac{2\kappa}{N}\sum_{k=1}^{N} \xi_k(\langle \psi_i, \psi_k \rangle + \langle \psi_k, \psi_i \rangle)(1 - \|\psi_i\|_2^2) \\ &= \frac{2\kappa}{N}\sum_{k=1}^{N} \xi_k \mathrm{Re}(\langle \psi_i, \psi_k \rangle)(1 - \|\psi_i\|_2^2). \end{aligned}$$

Then by Gronwall's lemma, we get

$$(\|\psi_i\|_2^2 - 1) = (\|\psi_i^0\|_2^2 - 1)\exp\left(\frac{2\kappa}{N}\sum_{k=1}^{N} \xi_k \int_{\mathbb{T}}^{d} \mathrm{Re}(\langle \psi_i, \psi_k \rangle)(s) ds\right).$$

Hence, since $\|\psi_i^0\|_2^2 = 1$, we obtain

$$\|\psi_i\|_2^2 = 1, \quad t > 0.$$

$\square$

Next, we show that the process of reducing the S-L system to the Kuramoto system [5]. Suppose that the system have constant one-body potentials and spatial homogeneous:

$$V_i(x) = \omega_i : \text{constant}, \quad \psi_i(x,t) := \psi_i(t), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+.$$

In this setting, the S-L system becomes

$$\mathrm{i}\dot{\psi}_i = \omega_i \psi_i + \frac{\mathrm{i}\kappa}{N} \sum_{k=1}^{N} \xi_k (\psi_k - \langle \psi_i, \psi_k \rangle \psi_i). \tag{2.2.5}$$

To derive the Kuramoto model from(2.2.5), we simply take the following ansatz for ψ_i:

$$\psi_i := e^{-\mathrm{i}\theta_i}, \quad 1 \le i \le N,$$

and substitute this ansatz into (2.2.5) to obtain

$$\dot{\theta}_i \psi_i = \omega_i \psi_i + \frac{\mathrm{i}\kappa}{N} \sum_{k=1}^{N} \xi_k (\psi_k - e^{-\mathrm{i}(\theta_i - \theta_k)} \psi_i).$$

Then, we take an inner product of the above relation with ψ_i, use the fact $|\psi_i(t)|^2 = 1$, and compare the real part of the resulting relation to get the Kuramoto model for classical synchronization [1, 5, 7, 8, 9, 10]:

$$\dot{\theta}_i = \omega_i + \frac{2\kappa}{N} \sum_{k=1}^{N} \xi_k \sin(\theta_k - \theta_i).$$

2.3 Review on wave function synchronization

In this section, we briefly review the previous results on the wave function for the S-L model with all-to-all cooperative couplings $\xi_k = 1$ for all k:

$$\mathrm{i}\partial_t \psi_i = -\Delta \psi_i + V_i(x)\psi_i + \frac{\mathrm{i}\kappa}{N} \sum_{k=1}^{N} (\psi_k - \langle \psi_i, \psi_k \rangle \psi_i). \tag{2.3.6}$$

As mentioned in Introduction, the S-L model (2.3.6) was first considered in Lohe's work [12] for the non-Abelian generalization of the Kuramoto model and its dynamics of the ensemble of coupled quantum Lohe oscillators has been studied in recent literature [2, 5, 6, 11, 13, 14]. So, in this section, we see the previous results from a recent perspective. First, for the wave function $\Psi = (\psi_1, \cdots, \psi_N)$ and one-body potential $\mathcal{V} = (V_1, \cdots, V_N)$, we introduce a

Lyapunov functionals for measuring the degree of quantum synchronization as

$$D(\Psi) := \max_{1\le i,j\le N} \|\psi_i - \psi_j\|_2, \quad D(\mathcal{V}) := \max_{1\le i,j\le N} \|V_i - V_j\|_\infty.$$

Then, we can stated the weak concept of synchronization in terms of $D(\Psi)$.

Definition 2.3.1. *[5, 6] Let $\Psi = (\psi_1, \cdots, \psi_N)$ be a solution to the S-L model* (1.0.1).

1. *The model* (1.0.1) *exhibits complete wave function synchronization if and only if the following estimate holds:*

$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\|_2 = c_{ij}, \quad 1 \le i, j \le N,$$

   where c_{ij} is a nonnegative constant.

2. *The model* (1.0.1) *exhibits practical wave function synchronization if and only if the following estimate holds:*

$$\lim_{K\to\infty} \limsup_{t\to\infty} \|\psi_i(t) - \psi_j(t)\|_2 = 0, \quad 1 \le i, j \le N.$$

For this definition, [5] derives the differential inequality:

$$\frac{d}{dt} D(\Psi)^2 \le \kappa \left[D(\Psi)^2 (2D(\Psi) - 1) + \frac{2D(\mathcal{V})}{\kappa} \right], \quad t > 0. \tag{2.3.7}$$

When (2.3.7) has same potential, it becomes as follows.

Lemma 2.3.1. *[5] Suppose that the coupling strength and one-body potential satisfy*

$$\kappa > 0, \quad D(\mathcal{V}) = 0.$$

Then, for any solution $\Psi = (\psi_1, \cdots, \psi_N)$ to (2.3.6)*, we have*

$$\frac{d}{dt} D(\Psi(t)) \le \kappa \left(2D(\Psi)^2 - D(\Psi) \right), \quad t > 0.$$

And then, Lemma 2.3.1 yields the quantum synchronization.

Theorem 2.3.1. *[5] Suppose that the coupling strength, one-body potentials and initial data satisfy*

$$\kappa > 0, \quad D(\mathcal{V}) = 0, \quad D(\Psi^0) < \frac{1}{2}.$$

Then, for any solution $\Psi = (\psi_1, \cdots, \psi_N)$ *to* (2.3.6)*, we have the complete synchronization:*

$$D(\Psi(t)) \leq \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{\kappa t}}, \quad t \geq 0.$$

Next, in [6] they assume that $D(\mathcal{V}) > 0$. Consider the cubic equation

$$f(x) := 2x^3 - x^2 + \frac{2D(\mathcal{V})}{\kappa} = 0, \quad x \in [0, \infty), \quad \kappa > 54D(\mathcal{V}). \tag{2.3.8}$$

Then, equation (2.3.8) has a positive local maximum $\frac{2D(\mathcal{V})}{\kappa}$ and a negative local minimum $\frac{2D(\mathcal{V})}{\kappa} - \frac{1}{27}$ at $x = 0$ and $\frac{1}{3}$, respectively. Moreover, (2.3.8) has two positive real roots, $\alpha_1 < \alpha_2$:

$$0 < \alpha_1 < \frac{1}{3} < \alpha_2 < \frac{1}{2}.$$

Clearly, the roots depend continuously on κ and $D(\mathcal{V})$, and

$$\lim_{\kappa \to \infty} \alpha_1 = 0, \quad \lim_{\kappa \to \infty} \alpha_2 = \frac{1}{2}.$$

Then, for $D(\mathcal{V})$, we have practical synchronization.

Theorem 2.3.2. *Suppose that the coupling strength, one-body potentials and initial data satisfy*

$$\kappa > 54D(\mathcal{V}) > 0, \quad D(\Psi^0) < D \quad for\,some\,positive\,constant\,D.$$

Then, for any solution $\Psi = (\psi_1, \cdots, \psi_N)$ *to* (2.3.6)*, we have the practical synchronization:*

$$\lim_{\kappa \to \infty} \limsup_{t \to \infty} D(\Psi(t)) = 0.$$

Chapter 3

A finite-dimensional reduction of the Schrödinger-Lohe system

In this Chapter, we introduce the pairwise correlation function h_{ij} and mention modified model treated in this thesis. And we also discuss the relationship between the S-L system and the L-V system.

First, we study a dynamical system for pairwise correlation function h_{ij}. For a given wave function $\Psi = (\psi_1, \cdots, \psi_N)$ whose component satisfies the S-L system (1.0.2), we introduce a complex-valued correlation function h_{ij}:

$$h_{ij}(t) := \int_{\mathbb{T}^d} \psi_i \bar{\psi}_j dx \ \in \ \mathbb{C}, \quad 1 \leq i, j \leq N. \tag{3.0.1}$$

Then, by definition of $h_{ij}, 1 \leq i, j \leq N$, we can easily see that

$$h_{ij} = \bar{h}_{ij}, \quad h_{ii} = 1, \quad |h_{ij}| = \left| \int_{\mathbb{T}^d} \psi_i \bar{\psi}_j dx \right| \leq \|\psi_i\|_2 \|\psi_j\|_2 = 1.$$

Remark 3.0.1. *For any two wave functions ψ_i and ψ_j with $\|\psi_i^0\|_2 = 1 = \|\psi_j^0\|_2$, we can easily think of the following relation:*

$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\|_2 = 0 \iff \lim_{t\to\infty} h_{ij} = 1.$$

So, for convenience, we consider the quantity of $1 - h_{ij}$ instead of h_{ij} in a synchronization phenomena of particles.

Proposition 3.0.1. *Let ψ_i be a solution to* (1.0.2). *Then h_{ij} defined in* (3.0.1) *satisfies the coupled system of ODEs as follows:*

$$\dot{h}_{ij} = -\mathrm{i}\omega_{ij}h_{ij} + \frac{\kappa}{N}\sum_{k=1}^{N}\xi_k(h_{ik}+h_{kj})(1-h_{ij}), \quad t>0, \quad 1\le i,j\le N, \tag{3.0.2}$$

where $\omega_{ij} := \omega_i - \omega_j$.

Proof. It follows from (1.0.2) that we have

$$(\mathrm{i}\partial_t\psi_i + \Delta\psi_i)\bar{\psi}_j = (V+\omega_i)\psi_i\bar{\psi}_j + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_k(\psi_k\bar{\psi}_j - \langle\psi_i,\psi_k\rangle\psi_i\bar{\psi}_j), \tag{3.0.3}$$

$$(\mathrm{i}\partial_t\psi_j + \Delta\psi_j)\bar{\psi}_i = (V+\omega_j)\psi_j\bar{\psi}_i + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_k(\psi_k\bar{\psi}_i - \langle\psi_j,\psi_k\rangle\psi_j\bar{\psi}_i). \tag{3.0.4}$$

The integral $\int_{\mathbb{T}^d}\Big((3.0.3) - \overline{(3.0.4)}\Big)\,dx$ leads to

$$\begin{aligned}
&\mathrm{i}\int_{\mathbb{T}^d}\partial_t(\psi_i\bar{\psi}_j)dx \\
&= \int_{\mathbb{T}^d}\left(\Delta\bar{\psi}_j\psi_i - \Delta\psi_i\bar{\psi}_j\right)dx + \int_{\mathbb{T}^d}(\omega_i-\omega_j)\psi_i\bar{\psi}_j dx \\
&\quad + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_k\int_{\mathbb{T}^d}(\psi_k\bar{\psi}_j - \langle\psi_i,\psi_k\rangle\psi_i\bar{\psi}_j)dx \\
&\quad + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_k\int_{\mathbb{T}^d}(\psi_i\bar{\psi}_k - \overline{\langle\psi_j,\psi_k\rangle}\psi_i\bar{\psi}_j)dx \\
&=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13} + \mathcal{I}_{14},
\end{aligned}$$

where we used the fact that $V_i = V_i(x)$ is a real-valued function. For $\mathcal{I}_{11}$, integrating by parts, then we get

$$\mathcal{I}_{11} = \int_{\mathbb{T}^d}\left(((\Delta\bar{\psi}_j)\psi_i - (\Delta\psi_i)\bar{\psi}_j\right)dx = \int_{\mathbb{T}^d}\left(\nabla\psi_i\nabla\bar{\psi}_j - \nabla\bar{\psi}_j\nabla\psi_i\right)dx = 0.$$

And for $\mathcal{I}_{12}$, $\mathcal{I}_{13}$ and $\mathcal{I}_{14}$, by definition of h_{ij}, we have

$$\mathcal{I}_{12} = \omega_{ij}h_{ij}, \quad \mathcal{I}_{13} = \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_K(h_{kj} - h_{ik}h_{ij}), \quad \mathcal{I}_{14} = \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N}\xi_K(h_{ik} - h_{kj}h_{ij}).$$

So we obtain the desired result. □

By definition of h_{ij}, we present the main system of this thesis. As mentioned in the previous Chapter 1, we assume that $a_{ik} = \xi_k$ for all i, k. In other words, only the force of the sending particles determines the communication weight in this network. And, by definition of h_{ij}, when $k = i$ we can easily see through $a_{ik}(h_{kj} - h_{ik}h_{ij}) = 0$ that a_{jj} are not related to the dynamics. So, in this assumption, let ψ_i be a solution to (1.0.1) with identical one-body potentials $V_i = V_j$. Then h_{ij} satisfies the following coupled S-L system:

$$\dot{h_{ij}} = \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k h_{ik} + \sum_{k\neq j}^{N} \xi_k h_{kj}\right)(1 - h_{ij}). \tag{3.0.5}$$

For this system, we discuss how the S-L model can be reduced to the L-V model following [14]. First, we set the real and imaginary parts of h_{ij} as R_{ij} and I_{ij}, respectively:

$$R_{ij} := Re(h_{ij}), \quad I_{ij} := Im(h_{ij}), \quad h_{ij} = R_{ij} + \mathrm{i}I_{ij}.$$

Then, the system (3.0.5) can be split into the ODE system for R_{ij} and I_{ij}:

$$\begin{aligned}
\dot{R_{ij}} &= \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k R_{ik} + \sum_{k\neq j}^{N} \xi_k R_{kj}\right)(1 - R_{ij}) \\
&\quad + \frac{\kappa}{N}\left(\sum_{k\neq i}^{N} \xi_k I_{ik} + \sum_{k\neq j}^{N} \xi_k I_{kj}\right) I_{ij}, \\
\dot{I_{ij}} &= \frac{\kappa}{N}\left(\sum_{k\neq i}^{N} \xi_k I_{ik} + \sum_{k\neq j}^{N} \xi_k I_{kj}\right)(1 - R_{ij}) \\
&\quad - \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k R_{ik} + \sum_{k\neq j}^{N} \xi_k R_{kj}\right) I_{ij}.
\end{aligned} \tag{3.0.6}$$

Because of the uniqueness of the ODE system (3.0.6), the set $\mathcal{I} := \{I_{ij} = 0,\ 1 \leq i < j \leq N\}$ is positively invariant for the dynamics (3.0.6). So, if we restrict (3.0.6) to the invariant set $\mathcal{I}$, then we get

$$\dot{R_{ij}} = \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k R_{ik} + \sum_{k\neq j}^{N} \xi_k R_{kj}\right)(1 - R_{ij}).$$

Thus, if we set $r_{ij} := 1 - R_{ij}$, then it satisfies

$$\dot{r_{ij}} = \frac{\kappa}{N} r_{ij} \left(\sum_{k \neq i}^{N} \xi_k r_{ik} + \sum_{k \neq j}^{N} \xi_k r_{kj} - 2 \sum_{k=1}^{N} \xi_k \right). \tag{3.0.7}$$

In fact, system (3.0.7) is generalized L-V system. For the case $\xi_i > 0$ for all i, we obtain cooperative L-V system. On the other hand, for the case $\xi_i < 0$ for all i, we obtain competitive L-V system.

Before closed this Chapter, we review the previous results on the coupled S-L system when it has pure communication weight. For this, we define $l_{1,\infty}$-like norm of $1 - h_{ij}(t)$ which has been introduced in [14]:

$$\mathcal{H}_i(t) := \sum_{k=1}^{N} |1 - h_{ik}(t)|, \quad \mathcal{H}(t) := \max_{1 \leq i \leq N} \mathcal{H}_i(t). \tag{3.0.8}$$

Then the emergent dynamics of the S-L model (3.0.5) can be implied by the following Theorem.

Theorem 3.0.1. *[14]*

1. *(Purely cooperative interactions) : Suppose that the coupling strength, initial data and interactive weights satisfy*

   $$\kappa > 0, \qquad \mathcal{H}^0 < N, \qquad \xi_k = 1 \quad for \quad 1 \leq k \leq N.$$

   Then, the solution $h_{ij}(t)$ to (3.0.5) *converges to 1 exponentially fast for all i and j.*

2. *(Purely competitive interactions) : Suppose that the coupling strength, initial data and interactive weights satisfy*

   $$\kappa > 0, \qquad \mathcal{H}^0 < N, \qquad \xi_k = -1 \quad for \quad 1 \leq k \leq N.$$

   Then, for any solution h_{ij} to (3.0.5)*, we have*

   $$\lim_{t \to \infty} \mathcal{H}(t) = N.$$

In other words, if all ξ_k have positive value, then all oscillators attract each other so that synchronization happens. Conversely, if all ξ_k have negative value, then all oscillators repeal each other, so synchronization does not happen. Then the question here is if ξ_k attain both positive and negative value, what dynamic pattern does h_{ij} have. We answer this question in next Chapter.

Chapter 4

Existence of equilibria for finite-dimension

In this Chapter, we present the existence of equilibria in a finite-dimension of coupled S-L system. In the first section, we show the existence of equilibria in a low-dimension, especially $N = 3$. And then, we generalize the dimension in the next section. Each section considers two cases: the total sum of communication weight is positive and negative. In particular, when the total sum is negative, we focus on the number of negative communication weights: only one, more than one and all. Throughout of this chapter, we restrict complex value h_{ij} on real line. Then we can say that $h_{ji} = h_{ij}$.
Recall the coupled S-L system:

$$\dot{h_{ij}} = \frac{\kappa}{N}\left(\xi_i + \xi_j + \sum_{k\neq i}^{N} \xi_k h_{ik} + \sum_{k\neq j}^{N} \xi_k h_{kj}\right)(1 - h_{ij}), \quad 1 \le i, j \le N. \tag{4.0.1}$$

Here, we assume that communication weight is nontrivial and strictly increasing, i.e. $\xi_k \neq 1$ for all k and $\xi_1 < \cdots < \xi_N$. And we set

$$\Xi := \sum_{k=1}^{N} \xi_k.$$

4.1 Low-dimensional system

In this section, we show the equilibria of coupled S-L system when $N = 3$. We focus on the value of Ξ. First, we assume that Ξ is positive. In this case, we show the complete synchronization. As mentioned in previous Chapter 3, we consider the the value of $1-h_{ij}$ instead of h_{ij} to show the synchronization of particles.

Lemma 4.1.1. *Let h_{ij} be the solutions of* (4.0.1)*. Then the quantity $1-h_{ij}$ satisfies the following differential inequality:*

$$\frac{d}{dt}|1-h_{ij}| \le \frac{\kappa}{3}\left(-2\Xi + \xi_3 \sum_{k\ne i}^{3} |1-h_{ik}| + \xi_3 \sum_{k\ne j}^{3} |1-h_{kj}|\right)|1-h_{ij}|.$$

Proof. It is enough considered h_{12}. Note that $1-h_{12}$ satisfies the following differential equation:

$$\frac{d}{dt}(1-h_{12}) = \frac{\kappa}{3}\left[-2\Xi + \sum_{k\ne 1}^{3} \xi_k(1-h_{1k}) + \sum_{k\ne 2}^{3} \xi_k(1-h_{k2})\right](1-h_{12}). \tag{4.1.1}$$

Multiply $\overline{1-h_{12}}$ to (4.1.1), then we have

$$\frac{1}{2}\frac{d}{dt}|1-h_{12}|^2 = \frac{\kappa}{3}\left[-2\Xi + \sum_{k\ne 1}^{3} \xi_k(1-h_{1k}) + \sum_{k\ne 2}^{3} \xi_k(1-h_{k2})\right]|1-h_{12}|^2.$$

Since $\xi_k \le \xi_3$ for all k by assumption, we have

$$\frac{1}{2}\frac{d}{dt}|1-h_{12}|^2 \le \frac{\kappa}{3}\left(-2\Xi + \xi_3 \sum_{k\ne 1}^{3} |1-h_{1k}| + \xi_3 \sum_{k\ne 2}^{3} |1-h_{k2}|\right)|1-h_{12}|^2,$$

and then we can find the desired differential inequality:

$$\frac{d}{dt}|1-h_{12}| \le \frac{\kappa}{3}\left(-2\Xi + \xi_3 \sum_{k\ne 1}^{3} |1-h_{1k}| + \xi_3 \sum_{k\ne 2}^{3} |1-h_{k2}|\right)|1-h_{12}|.$$

Similarly, for all $1 \le i < j \le 3$, we obtain the desired inequality. □

Then, by Lemma 4.1.1, we can show the complete synchronization, which is the first main theorem of this section.

Theorem 4.1.1. *Suppose that the coupling strength, communication weight and initial data satisfy*

$$\kappa > 0, \quad \Xi > 0, \quad \sum_{i,j=1}^{3} |1 - h_{ij}^0| \le \frac{\Xi}{\xi_3}.$$

Then, the solution h_{ij} to (4.0.1) *converges to 1 exponentially fast for all* $1 \le i < j \le 3$.

Proof. By Lemma 4.1.1, we obtain

$$\begin{aligned}
\sum_{i<j}^{3} \frac{d}{dt}|1 - h_{ij}| &= \frac{d}{dt}\sum_{i<j}^{3}|1 - h_{ij}| \\
&\le \sum_{i<j}^{3}\left[\frac{\kappa}{3}\left(-2\Xi + \xi_3\sum_{k\neq i}^{3}|1 - h_{ik}| + \xi_3\sum_{k\neq j}^{3}|1 - h_{kj}|\right)|1 - h_{ij}|\right] \\
&\le \frac{2\kappa}{3}\left(-\Xi + \xi_3\sum_{i<j}^{3}|1 - h_{ij}|\right)\sum_{i<j}^{3}|1 - h_{ij}|,
\end{aligned}$$

which is the Riccai differential inequality. Therefore, we get

$$\sum_{i<j}^{3}|1 - h_{ij}| \le \left[\left(\frac{1}{\sum_{i<j=1}^{3}|1 - h_{ij}^0|} - \frac{\xi_3}{\Xi}\right)e^{\frac{2\kappa}{3}\Xi t} + \frac{\xi_3}{\Xi}\right]^{-1},$$

where the initial data satisfy

$$\sum_{i<j=1}^{3}|1 - h_{ij}^0| \le \frac{\Xi}{\xi_3}.$$

This leads to the desired result. □

Now, we assume that Ξ is negative. In this case, we focus on the number of negative communication weights: only one, two and three. So first, we assume that the system has only one negative communication weight, i.e. $\xi_1 < 0 < \xi_2 < \cdots < \xi_N$. For convenience of notation, we define

$$X := 1 + h_{12}, \quad Y := 1 + h_{13}, \quad Z := 1 - h_{23}.$$

Then, we get the following Lemma.

Lemma 4.1.2. *Let h_{ij} be the solutions of* (4.0.1). *Then (X, Y, Z) satisfies the following differential inequality:*

$$\begin{aligned}
\frac{d}{dt}|X| &\le \frac{\kappa}{3}\left[2(\xi_1+\xi_2) - (\xi_1+\xi_2)|X| + \xi_3|Y| + \xi_3|Z|\right]|X| + \frac{2\kappa\xi_3}{3}\left(|Y|+|Z|\right),\\
\frac{d}{dt}|Y| &\le \frac{\kappa}{3}\left[2(\xi_1+\xi_3) - (\xi_1+\xi_3)|Y| + \xi_2|X| + \xi_2|Z|\right]|Y| + \frac{2\kappa\xi_2}{3}\left(|X|+|Z|\right),\\
\frac{d}{dt}|Z| &\le \frac{\kappa}{3}\left[2(\xi_1-\xi_2-\xi_3) + (\xi_2+\xi_3)|Z| - \xi_1|X| + \xi_1|Y|\right]|Z|.
\end{aligned} \tag{4.1.2}$$

Proof. First, h_{12} satisfies the following differential equation:

$$\dot{h_{12}} = \frac{\kappa}{3}\left[(\xi_1+\xi_2)(1+h_{12}) + \xi_3 h_{13} + \xi_3 h_{23}\right](1-h_{12}).$$

Then, by definition of X, we get

$$\frac{d}{dt}X = \frac{\kappa}{3}[(\xi_1+\xi_2)X + \xi_3 Y - \xi_3 Z](2-X). \tag{4.1.3}$$

So if we multiply $\overline{X}$ to (4.1.3), then we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|X|^2 &= \frac{2\kappa}{3}[(\xi_1+\xi_2)X + \xi_3 Y - \xi_3 Z]\overline{X} - \frac{\kappa}{3}[(\xi_1+\xi_2)X + \xi_3 Y - \xi_3 Z]|X|^2\\
&:= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

Then by assumption, we have

$$\mathcal{I}_1 = \frac{2\kappa}{3}[(\xi_1+\xi_2)X + \xi_3 Y - \xi_3 Z]\overline{X} \le \frac{2\kappa}{3}[(\xi_1+\xi_2)|X| + \xi_3|Y| + \xi_3|Z|]|X|,$$

and

$$\mathcal{I}_2 = \frac{\kappa}{3}[(\xi_1+\xi_2)X + \xi_3 Y - \xi_3 Z]|X|^2 \le \frac{\kappa}{3}[(\xi_1+\xi_2)|X| + \xi_3|Y| + \xi_3|Z|]|X|^2.$$

Hence, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|X|^2 \\
&\leq \frac{\kappa}{3}[2(\xi_1+\xi_2)-(\xi_1+\xi_2)|X|+\xi_3|Y|+\xi_3|Z|]|X|^2+\frac{2\kappa\xi_3}{3}\left(|Y|+|Z|\right)|X|,
\end{aligned}$$

and then we can find the desired differential inequality:

$$\frac{d}{dt}|X| \leq \frac{\kappa}{3}\left[2(\xi_1+\xi_2)-(\xi_1+\xi_2)|X|+\xi_3|Y|+\xi_3|Z|\right]|X|+\frac{2\kappa\xi_3}{3}\left(|Y|+|Z|\right).$$

Similarly, we obtain the desired inequality. □

Then, by Lemma 4.1.2, we can show the second main results.

Theorem 4.1.2. *Suppose that the coupling strength and communication weight satisfy*

$$\kappa > 0, \quad \xi_1 < 0 < \xi_2 < \xi_3, \quad \Xi < 0.$$

If communication weight and initial data satisfy

$$\xi_1 < -2\xi_3, \quad |X^0|+|Y^0|+|Z^0| \leq \frac{\xi_1+2\xi_3}{\xi_1},$$

then, for any solution h_{ij} *to* (4.0.1)*, there exists a positive constant* C *such that*

$$|X|+|Y|+|Z| \leq e^{-Ct} \quad as \quad t \to \infty.$$

In other words, ψ_2 and ψ_3 are moved away from ψ_1 by ξ_1, and then ψ_2 and ψ_3 are synchronized. So that, h_{12} and h_{13} converge to -1 and h_{23} converges to 1. Hence Theorem 4.1.2 shows bi-polar synchronization.

Proof. First, if we add the differential inequalities in (4.1.2), then we have

$$\begin{aligned}
&\frac{d}{dt}(|X|+|Y|+|Z|) \\
&\leq \frac{\kappa}{3}\left[(2(\xi_1+2\xi_2)-(\xi_1+\xi_2)|X|+\xi_3|Y|+\xi_3|Z|\right]|X| \\
&\quad+\frac{\kappa}{3}\left[(2(\xi_1+2\xi_3)+\xi_2|X|-(\xi_1+\xi_3)|Y|+\xi_2|Z|\right]|Y| \\
&\quad+\frac{\kappa}{3}\left[2\xi_1-\xi_1|X|-\xi_1|Y|+(\xi_2+\xi_3)|Z|\right]|Z|.
\end{aligned}$$

Since we assume that $\xi_1 < \xi_1 + 2\xi_2 < \xi_1 + 2\xi_3 < 0$ and $\xi_k \le -\xi_1$ for all k, we get

$$\begin{aligned}
&\frac{d}{dt}(|X|+|Y|+|Z|)\\
&\le \frac{\kappa}{3}\left[(2(\xi_1+2\xi_3) - 2\xi_1|X| - \xi_1|Y| - \xi_1|Z|)\right]|X|\\
&\quad + \frac{\kappa}{3}\left[(2(\xi_1+2\xi_3) - \xi_1|X| - 2\xi_1|Y| - \xi_1|Z|)\right]|Y|\\
&\quad + \frac{\kappa}{3}\left[(2(\xi_1+2\xi_3) - \xi_1|X| - \xi_1|Y| - 2\xi_1|Z|)\right]|Z|\\
&\le \frac{2\kappa}{3}\left[(\xi_1+2\xi_2) - \xi_1(|X|+|Y|+|Z|)\right](|X|+|Y|+|Z|),
\end{aligned}$$

which is the Riccati differential inequality. So, we obtain

$$\begin{aligned}
&(|X|+|Y|+|Z|)\\
&\le \left[\left(\frac{1}{(|X^0|+|Y^0|+|Z^0|)} - \frac{\xi_1}{\xi_1+2\xi_3}\right)e^{-\frac{2\kappa}{2}(\xi_1+2\xi_3)t} + \frac{\xi_1}{\xi_1+2\xi_3}\right]^{-1},
\end{aligned}$$

where initial data satisfy

$$|X^0|+|Y^0|+|Z^0| \le \frac{\xi_1+2\xi_3}{\xi_1}.$$

So we can completes the proof. $\square$

In the same way, when the system have two negative communication weight: $\xi_1 < \xi_2 < 0 < \xi_3$, we can also show the bi-polar synchronization.

Theorem 4.1.3. *Suppose that the coupling strength and communication weight satisfy*

$$\kappa > 0, \quad \xi_1 < \xi_2 < 0 < \xi_3, \quad \xi_1 < -2\xi_3, \quad \xi_1 < 2\xi_2, \quad \Xi < 0.$$

If the communication weight and initial data satisfy

$$\xi_2 + \xi_3 > 0, \quad |X^0|+|Y^0|+|Z^0| \le \frac{\xi_1+2\xi_3}{\xi_1},$$

either

$$\xi_2 + \xi_3 < 0, \quad |X^0|+|Y^0|+|Z^0| \le \frac{\xi_1-2\xi_2}{\xi_1},$$

then, for any solution h_{ij} to (4.0.1), *there exists a positive constant C such that*

$$|X| + |Y| + |Z| \le e^{-Ct} \quad as \quad t \to \infty.$$

Proof. In this case, the process is similar to the above way, So we will look briefly. First, by definition of X, Y and Z, we have

$$\begin{aligned}
&\frac{d}{dt}(|X| + |Y| + |Z|) \\
&\le \frac{\kappa}{3}[2\xi_1 - (\xi_1 + \xi_2)|X| + \xi_3|Y| + \xi_3|Z|]|X| \\
&\quad + \frac{\kappa}{3}[2(\xi_1 + 2\xi_3) - \xi_2|X| - (\xi_1 + \xi_3)|Y| - \xi_2|Z|]|Y| \\
&\quad + \frac{\kappa}{3}[2(\xi_1 - 2\xi_2) - \xi_1|X| - \xi_1|Y| + (-\xi_2 + \xi_3)|Z|]|Z|.
\end{aligned}$$

If $\xi_2 + \xi_3 > 0$, then $\xi_1 < \xi_1 - 2\xi_2 < \xi_1 + 2\xi_3 < 0$. So that we have

$$\frac{d}{dt}(|X| + |Y| + |Z|) \le \frac{2\kappa}{3}[\xi_1 + 2\xi_3 - \xi_1(|X| + |Y| + |Z|)](|X| + |Y| + |Z|),$$

which is the Riccati differential inequality. Hence we obtain

$$\begin{aligned}
&(|X| + |Y| + |Z|) \\
&\le \left[\left(\frac{1}{|X|^0 + |Y|^0 + |Z|^0} - \frac{\xi_1 + 2\xi_3}{\xi_1}\right) e^{\frac{2\kappa}{3}(\xi_1 + 2\xi_3)t} + \frac{\xi_1 + 2\xi_3}{\xi_1}\right]^{-1},
\end{aligned}$$

where the initial data satisfy

$$|X^0| + |Y^0| + |Z^0| \le \frac{\xi_1 + 2\xi_3}{\xi_1}.$$

If $\xi_2 + \xi_3 < 0$, then $\xi_1 < \xi_1 + 2\xi_3 < \xi_1 - 2\xi_2 < 0$. So that we have

$$\frac{d}{dt}(|X| + |Y| + |Z|) \le \frac{2\kappa}{3}[\xi_1 - 2\xi_2 - \xi_1(|X| + |Y| + |Z|)](|X| + |Y| + |Z|),$$

which is the Riccati differential inequality. Hence we obtain

$$\begin{aligned}
&(|X| + |Y| + |Z|) \\
&\le \left[\left(\frac{1}{|X|^0 + |Y|^0 + |Z|^0} - \frac{\xi_1 - 2\xi_2}{\xi_1}\right) e^{\frac{2\kappa}{3}(\xi_1 - 2\xi_2)t} + \frac{\xi_1 - 2\xi_2}{\xi_1}\right]^{-1},
\end{aligned}$$

where the initial data satisfy

$$|X^0| + |Y^0| + |Z^0| \leq \frac{\xi_1 - 2\xi_2}{\xi_1}.$$

Therefore, we have the desired results. $\square$

Finally, we assume that all communication weights are negative: $\xi_1 < \xi_2 < \xi_3 < 0$. Indeed, in this case, all the particles repeal each other. But if we give proper assumption, we can show the bi-polar synchronization.

Theorem 4.1.4. *Suppose that the coupling strength, communication weight and initial data satisfy*

$$\kappa > 0,\ \xi_1 < \xi_2 < \xi_3 < 0,\ \xi_1 < 2(\xi_2+\xi_3),\ |X^0|+|Y^0|+|Z^0| \leq \frac{\xi_1 - 2(\xi_2+\xi_3)}{\xi_1}.$$

Then, for any solution h_{ij} to (4.0.1)*, there exists a positive constant C such that*

$$|X| + |Y| + |Z| \leq e^{-Ct} \quad as \quad t \to \infty.$$

Proof. The proof is similar to the above way. So we will briefly explain. Since we assume that $\xi_k < 0$ for all k and $\xi_1 < 2(\xi_2 + \xi_3) < 0$, we have

$$\begin{aligned}
&\frac{d}{dt}(|X| + |Y| + |Z|)\\
&\leq \frac{2\kappa}{3}[\xi_1 - 2(\xi_2+\xi_3) - \xi_1(|X| + |Y| + |Z|)](|X| + |Y| + |Z|),
\end{aligned}$$

which is the Riccati differential inequality. Hence we obtain

$$\begin{aligned}
&|X| + |Y| + |Z|\\
&\leq \Bigg[\left(\frac{1}{|X^0| + |Y^0| + |Z^0|} - \frac{\xi_1}{\xi_1 - 2(\xi_2+\xi_3)}\right)e^{-\frac{2\kappa}{3}(\xi_1 - 2(\xi_2+\xi_3))t}\\
&\qquad\qquad + \frac{\xi_1}{\xi_1 - 2(\xi_2+\xi_3)}\Bigg]^{-1},
\end{aligned}$$

where the initial data satisfy

$$|X^0| + |Y^0| + |Z^0| \leq \frac{\xi_1 - 2(\xi_2+\xi_3)}{\xi_1}.$$

So we can get the desired results. $\square$

Remark 4.1.1. *In this section, we find that if ξ_1 is much smaller than ξ_2 and ξ_3, then ξ_1 dominates the network of the system. So in next section, we will generalize this idea.*

4.2 Large-dimensional system

In this section, we show the equilibria of coupled S-L system in large-dimension. As before, we focus on the value of Ξ. First we assume that Ξ is positive. In this case, we show the complete synchronization.

Lemma 4.2.1. *Let h_{ij} be the solutions of* (4.0.1)*. Then the quantitiy $1-h_{ij}$ satisfies the following differential inequality:*

$$\frac{d}{dt}|1-h_{ij}| \le \frac{\kappa}{N}\left(-2\Xi + \xi_N \sum_{k\neq i}^{N} |1-h_{ik}| + \xi_N \sum_{k\neq j}^{N} |1-h_{kj}|\right)|1-h_{ij}|.$$

Proof. Since $\xi_k \le \xi_N$ for all k by assumption, we can apply the same argument in Lemma 4.1.1. Then we obtain the desired differential inequality. $\square$

For the next lemma, we use the $\mathcal{H}_i(t)$ and $\mathcal{H}(t)$ mentioned in the previous Chapter 3.

Lemma 4.2.2. *Let h_{ij} be the solutions of* (4.0.1)*. Then $\mathcal{H}$ satisfies*

$$\dot{\mathcal{H}} \le -\frac{2\kappa\xi}{N}\mathcal{H} + \frac{2\kappa\xi_N}{N}\mathcal{H}^2.$$

Proof. By definition of $\mathcal{H}_i$ and $\mathcal{H}$, h_{ij} satisfies

$$\begin{aligned}\frac{d}{dt}|1-h_{ij}| &\le \frac{\kappa}{N}\left(-2\Xi + \xi_N \sum_{k\neq i}^{N} |1-h_{ik}| + \xi_N \sum_{k\neq j}^{N} |1-h_{kj}|\right)|1-h_{ij}| \\ &\le \frac{\kappa}{N}\left[-2\Xi + \xi_N(\mathcal{H}_i + \mathcal{H}_j)\right]|1-h_{ij}| \\ &\le \frac{2\kappa}{N}\left(\Xi + \xi_N\mathcal{H}\right)|1-h_{ij}|.\end{aligned}$$

Sum up above relation over all indices j, then we have

$$\dot{\mathcal{H}}_i \le \frac{2\kappa}{N}\left(\Xi + \xi_N\mathcal{H}\right)\mathcal{H}_i. \tag{4.2.1}$$

Let i_t be the extremal index such that

$$\mathcal{H}(t) = \mathcal{H}_{i_t}(t).$$

With such extremal index i_t and (4.2.1), we have

$$\dot{\mathcal{H}} \le \frac{2\kappa}{N} \left(\Xi + \xi_N \mathcal{H}\right) \mathcal{H}.$$

$\square$

Then, by Lemma 4.2.1 and Lemma 4.2.2, we can show the complete synchronization.

Theorem 4.2.1. *Suppose that the coupling strength, communication weight and initial data satisfy*

$$\kappa > 0, \quad \Xi > 0, \quad \mathcal{H}^0 \le \frac{\Xi}{\xi_N}.$$

Then, the solution h_{ij} to (4.0.1) *converges to 1 exponentially fast for all i and j.*

Proof. As a result of Lemma 4.2.2, we have

$$\dot{\mathcal{H}} \le \frac{2\kappa}{N} \left(\Xi + \xi_N \mathcal{H}\right) \mathcal{H}.$$

which is the Riccati differential inequality. So we get

$$\mathcal{H}(t) \le \left[\left(\frac{1}{\mathcal{H}^0} - \frac{\xi_N}{\Xi} \right) e^{\frac{2\kappa}{N}\Xi t} + \frac{\xi_N}{\Xi} \right]^{-1},$$

where the initial data satisfy

$$\mathcal{H}^0 \le \frac{\Xi}{\xi_N}.$$

Therefore we obtain the desired result. $\square$

Now, we assume that Ξ is negative. In this case, we focus on the number of negative communication weight: only one, more than one and all. First, we assume that the system has only one negative communication: $\xi_1 < 0 < \xi_2 < \cdots < \xi_N$. For convenience of notation, we define

$$X_l := 1 + h_{1l}, \quad Y_{lm} := 1 - h_{lm}, \quad 2 \le l, m \le N.$$

Then we can show the quantity of (X_l, Y_{lm}).

Lemma 4.2.3. *Let h_{ij} be the solutions of* (4.0.1)*. Then (X_l, Y_{lm}) satisfies the following differential inequality:*

$$\begin{aligned}
\frac{d}{dt}|X_l| &\le \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\sum_{k\ne 1}^{N}|X_k| + \xi_2\sum_{k\ne 1,l}^{N}|Y_{kl}|\right) \\
&\quad + \frac{\kappa}{N}\left(-\xi_1|X_l| + \xi_2\sum_{k\ne 1}^{N}|X_k| + \xi_N\sum_{k\ne 1,l}^{N}|Y_{kl}|\right)|X_l|, \\
\frac{d}{dt}|Y_{lm}| &\le \frac{\kappa}{N}\left[2\left(\xi_1 - \sum_{k\ne 1}^{N}\xi_k\right) - \xi_1(|X_l| + |X_m|)\right]|Y_{lm}| \\
&\quad + \frac{\kappa\xi_N}{N}\left(\sum_{k\ne 1,l}^{N}|Y_{lk}| + \sum_{k\ne 1,m}^{N}|Y_{km}|\right)|Y_{lm}|.
\end{aligned}$$

Proof. First, h_{1l} satisfies the following differential equation:

$$\dot{h_{1l}} = \frac{\kappa}{N}\left[\xi_1(1 + h_{1l}) + \xi_l + \sum_{k\ne 1}^{N}\xi_k h_{1k} + \sum_{k\ne 1,l}^{N}\xi_k h_{kl}\right](1 - h_{1l}).$$

Then, by definition of X_l, we get

$$\frac{d}{dt}X_l = \frac{\kappa}{N}\left(\xi_1 X_l + \sum_{k\ne 1}^{N}\xi_k X_k - \sum_{k\ne 1,l}^{N}\xi_k Y_{kl}\right)(2 - X_l). \tag{4.2.2}$$

If we multiply $\overline{X}_1$ to (4.2.2), then we obtain

$$\frac{1}{2}\frac{d}{dt}|X_l|^2 = \frac{2\kappa}{N}\left(\xi_1 X_l + \sum_{k\ne 1}^{N}\xi_k X_k - \sum_{k\ne 1,l}^{N}\xi_k Y_{kl}\right)\overline{X}_l$$

$$
\begin{aligned}
&- \frac{\kappa}{N}\left(\xi_1 X_l + \sum_{k\neq 1}^{N} \xi_k X_k - \sum_{k\neq 1,l}^{N} \xi_k Y_{kl}\right)|X_l|^2 \\
&:= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

So that, by assumption, we have

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{2\kappa}{N}\left(\xi_1 X_l + \sum_{k\neq 1}^{N} \xi_k X_k - \sum_{k\neq 1,l}^{N} \xi_k Y_{kl}\right)\overline{X}_l \\
&\le \frac{2\kappa}{N}\left(\xi_1 |X_l| + \xi_N \sum_{k\neq 1}^{N} |X_k| + \xi_2 \sum_{k\neq 1,l}^{N} |Y_{kl}|\right)|X_l|,
\end{aligned} \tag{4.2.3}
$$

and

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{\kappa}{N}\left(-\xi_1 X_l - \sum_{k\neq 1}^{N} \xi_k X_k + \sum_{k\neq 1,l}^{N} \xi_k Y_{kl}\right)|X_l|^2 \\
&\le \frac{\kappa}{N}\left(-\xi_1 |X_l| + \xi_2 \sum_{k\neq 1}^{N} |X_k| + \xi_N \sum_{k\neq 1,l}^{N} |Y_{kl}|\right)|X_l|^2.
\end{aligned} \tag{4.2.4}
$$

So, if we combine (4.2.3) and (4.2.4), then we have

$$
\begin{aligned}
\frac{d}{dt}|X_l| \le \frac{2\kappa}{N}&\left(\xi_1 |X_l| + \xi_N \sum_{k\neq 1}^{N} |X_k| + \xi_2 \sum_{k\neq 1,l}^{N} |Y_{kl}|\right) \\
&+ \frac{\kappa}{N}\left(-\xi_1 |X_l| + \xi_2 \sum_{k\neq 1}^{N} |X_k| + \xi_N \sum_{k\neq 1,l}^{N} |Y_{kl}|\right)|X_l|.
\end{aligned}
$$

In the similar way, we obtain

$$
\begin{aligned}
\frac{d}{dt}|Y_{lm}| \le \frac{\kappa}{N}&\left[2\left(\xi_1 - \sum_{k\neq 1}^{N} \xi_k\right) - \xi_1(|X_l| + |X_m|)\right]|Y_{lm}| \\
&+ \frac{\kappa \xi_N}{N}\left(\sum_{k\neq 1,l}^{N} |Y_{lk}| + \sum_{k\neq 1,m}^{N} |Y_{km}|\right)|Y_{lm}|.
\end{aligned}
$$

□

Here, we introduce the $l_{1,\infty}$-like norm of X_k and Y_{kl}, respectively:

$$\mathcal{G}_1(t) := \sum_{k\neq 1}^N |X_k|, \quad \tilde{\mathcal{G}}_l(t) := \sum_{k\neq 1}^N |Y_{lk}|, \quad \mathcal{G}_2(t) := \max_{l\neq 1} \tilde{\mathcal{G}}_l(t), \quad t>0.$$

Then, by definition of $\mathcal{G}_1, \mathcal{G}_l$ and $\mathcal{G}_2$, we can show the following Lemma.

Lemma 4.2.4. *Let h_{ij} be the solution of* (4.0.1)*. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the following inequality:*

$$\frac{d}{dt}\mathcal{G}_1 \le \frac{2\kappa}{N}\left[\xi_1\mathcal{G}_1 + (N-1)(\xi_N\mathcal{G}_1 + \xi_2\mathcal{G}_2)\right] + \frac{\kappa}{N}\left[(-\xi_1+\xi_2)\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]\mathcal{G}_1,$$
$$\frac{d}{dt}\mathcal{G}_2 \le \frac{2\kappa}{N}\left[(\xi_1 - \sum_{k\neq 1}^N \xi_k) - \xi_1\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]\mathcal{G}_2.$$

Proof. By Lemma 4.2.3, X_l satisfy

$$\begin{aligned}\frac{d}{dt}|X_l| &\le \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\sum_{k\neq 1}^N |X_k| + \xi_2\sum_{k\neq 1,l}^N |Y_{kl}|\right)\\ &\quad + \frac{\kappa}{N}\left(-\xi_1|X_l| + \xi_2\sum_{k\neq 1}^N |X_k| + \xi_N\sum_{k\neq 1,l}^N |Y_{kl}|\right)|X_l|\\ &:= \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Then, by assumption, we can see that

$$\begin{aligned}\mathcal{I}_1 &= \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\sum_{k\neq 1}^N |X_k| + \xi_2\sum_{k\neq 1,l}^N |Y_{kl}|\right)\\ &\le \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\mathcal{G}_1 + \xi_2\tilde{\mathcal{G}}_l\right)\\ &\le \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\mathcal{G}_1 + \xi_2\mathcal{G}_2\right),\end{aligned} \tag{4.2.5}$$

and

$$\begin{aligned}\mathcal{I}_2 &= \frac{\kappa}{N}\left(-\xi_1 X_l + \xi_2\sum_{k\neq 1}^N |X_k| + \xi_N\sum_{k\neq 1,l}^N |Y_{kl}|\right)|X_l|\\ &\le \frac{\kappa}{N}\left(-\xi_1\mathcal{G}_1 + \xi_2\mathcal{G}_1 + \xi_N\tilde{\mathcal{G}}_l\right)|X_l|\\ &\le \frac{\kappa}{N}\left[(-\xi_1+\xi_2)\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]|X_l|.\end{aligned} \tag{4.2.6}$$

So, if we combine (4.2.5) and (4.2.6), then we have

$$\frac{d}{dt}|X_l| \leq \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_N\mathcal{G}_1 + \xi_2\mathcal{G}_2\right) + \frac{\kappa}{N}\left[(-\xi_1+\xi_2)\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]|X_l|. \tag{4.2.7}$$

Sum up (4.2.7) over all indies $l \neq 1$, then we have

$$\frac{d}{dt}\mathcal{G}_1 \leq \frac{2\kappa}{N}\left[\xi_1\mathcal{G}_1 + (N-1)(\xi_N\mathcal{G}_1 + \xi_2\mathcal{G}_2)\right] + \frac{\kappa}{N}\left[(-\xi_1+\xi_2)\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]\mathcal{G}_1.$$

In a similar way, for Y_{lm}, we get

$$\frac{d}{dt}\tilde{\mathcal{G}}_l \leq \frac{2\kappa}{N}\left[(\xi_1 - \sum_{k\neq 1}^{N}\xi_k) - \xi_1\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]\tilde{\mathcal{G}}_l. \tag{4.2.8}$$

Let l_t be the extremal index such that

$$\mathcal{G}_2(t) = \mathcal{G}_{l_t}(t).$$

With such extremal index l_t and (4.2.5), we have

$$\frac{d}{dt}\mathcal{G}_2 \leq \frac{2\kappa}{N}\left[\left(\xi_1 - \sum_{k\neq 1}^{N}\xi_k\right) - \xi_1\mathcal{G}_1 + \xi_N\mathcal{G}_2\right]\mathcal{G}_2.$$

□

Then, by Lemma 4.2.3 and 4.2.4, we get the following Theorem.

Theorem 4.2.2. *Suppose that the coupling strength and communication weight satisfy*

$$\kappa > 0, \quad \xi_1 < 0 < \xi_2 < \cdots < \xi_N, \quad \Xi < 0.$$

If communication weight and initial data satisfy

$$\xi_1 + (N-1)\xi_N < 0, \quad \mathcal{G}_1^0 + \mathcal{G}_2^0 \leq \frac{\xi_1 + (N-1)\xi_N}{\xi_1},$$

then, for any solution h_{ij} to (4.0.1), there exists a positive constant C such that

$$\mathcal{G}_1 + \mathcal{G}_2 \leq e^{-Ct} \quad as \quad t \to \infty.$$

In other words, $\psi_2, \cdots, \psi_N$ are move away from ψ_1 by ξ_1, and then $\psi_2, \cdots, \psi_N$ are synchronized. So that h_{1l} converges to -1 and h_{lm} converges to 1. So we can show the bi-polar synchronization.

Proof. First, by Lemma 4.2.4, we obtain

$$\begin{aligned}\frac{d}{dt}(\mathcal{G}_1+\mathcal{G}_2) \le \frac{\kappa}{N}\left[2(\xi_1+(N-1)\xi_N)+(-\xi_1+\xi_2)\mathcal{G}_1+\xi_N\mathcal{G}_2\right]\mathcal{G}_1 \\ +\frac{\kappa}{N}\left[2\left(\xi_1-\sum_{k\neq 1}^{N}\xi_k+(N-1)\xi_2\right)-2\xi_1\mathcal{G}_1+2\xi_N\mathcal{G}_2\right]\mathcal{G}_2.\end{aligned}$$

Then, since $\xi_1-\sum_{k\neq 1}^{N}\xi_k+(N-1)\xi_2<\xi_1+(N-1)\xi_N<0$ and $\xi_k\le-\xi_1$ for all k by assumption, we get

$$\frac{d}{dt}(\mathcal{G}_1+\mathcal{G}_2)\le\frac{2\kappa}{N}\left[(\xi_1+(N-1)\xi_N)-\xi_1(\mathcal{G}_1+\mathcal{G}_2)\right](\mathcal{G}_1+\mathcal{G}_2),$$

which is the Riccati differential inequality. So we obtain

$$\begin{aligned}&(\mathcal{G}_1+\mathcal{G}_2)\\ &\le\left[\left(\frac{1}{\mathcal{G}_1^0+\mathcal{G}_2^0}-\frac{\xi_1}{\xi_1+(N-1)\xi_N}\right)e^{-\frac{2\kappa}{N}(\xi_1+(N-1)\xi_N)t}+\frac{\xi_1}{\xi_1+(N-1)\xi_N}\right]^{-1},\end{aligned}$$

where the initial data satisfy

$$\mathcal{G}_1^0+\mathcal{G}_2^0\le\frac{\xi_1+(N-1)\xi_N}{\xi_1}.$$

Therefore, we get the desired result. □

Now, we assume that the system has more than one negative communication weight. Then there exist n such that $n\neq 1, N$ and $\xi_1<\cdots<\xi_n<0<\xi_{n+1}<\cdots<\xi_N$. For next Theorem, we define

$$\xi_M:=\max_{k\neq 1}|\xi_k|>0.$$

Then, similar way, we can show the bi-polar synchronization.

Theorem 4.2.3. *Suppose that there exist $n \neq 1, N$ such that*

$$\xi_1 < \cdots < \xi_n < 0 < \xi_{n+1} < \cdots < \xi_N,$$

and the coupling strength, communication weight and initial data satisfy

$$\kappa > 0, \quad \tilde{\xi} < 0, \quad \Xi < 0, \quad \mathcal{G}_1^0 + \mathcal{G}_2^0 \leq \frac{\tilde{\xi}}{\xi_1},$$

where

$$\tilde{\xi} := \max\{\xi_1 + (N-1)\xi_M, \xi_1 - \sum_{k\neq 1}^{N} \xi_k - (N-1)\xi_2\}.$$

Then, for any solution h_{ij} to (4.0.1)*, there exists a positive constant C such that*

$$\mathcal{G}_1 + \mathcal{G}_2 \leq e^{-Ct} \quad as \quad t \to \infty.$$

Proof. Using the definition of ξ_M, Lemma 4.2.3 is rewritten as:

$$\begin{aligned}
\frac{d}{dt}|X_l| &\leq \frac{2\kappa}{N}\left(\xi_1|X_l| + \xi_M \sum_{k\neq 1}^{N}|X_k| - \xi_2 \sum_{k\neq 1,l}^{N}|Y_{kl}|\right) \\
&\quad + \frac{\kappa}{N}\left(-\xi_1|X_l| - \xi_2\sum_{k\neq 1}^{N}|X_k| + \xi_M \sum_{k\neq 1,l}^{N}|Y_{kl}|\right)|X_l|, \\
\frac{d}{dt}|Y_{lm}| &\leq \frac{\kappa}{N}\left[2\left(\xi_1 - \sum_{k\neq 1}^{N}\xi_k\right) - \xi_1(|X_l| + |X_m|)\right]|Y_{lm}| \\
&\quad + \frac{\kappa\xi_M}{N}\left(\sum_{k\neq 1,l}^{N}|Y_{lk}| + \sum_{k\neq 1,m}^{N}|Y_{km}|\right)|Y_{lm}|.
\end{aligned}$$

And then, by definition of $\mathcal{G}_1, \mathcal{G}_2$, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}_1 &\leq \frac{2\kappa}{N}\left[\xi_1 + (N-1)(\xi_M\mathcal{G}_1 - \xi_2\mathcal{G}_2)\right] + \frac{\kappa}{N}\left[(-\xi_1 - \xi_2)\mathcal{G}_1 + \xi_M\mathcal{G}_2\right]\mathcal{G}_1, \\
\frac{d}{dt}\mathcal{G}_2 &\leq \frac{2\kappa}{N}\left[\left(\xi_1 - \sum_{k\neq 1}^{N}\xi_k\right) - \xi_1\mathcal{G}_1 + \xi_M\mathcal{G}_2\right]\mathcal{G}_2.
\end{aligned}$$

If we set

$$\tilde{\xi} := \max\{\xi_1 + (N-1)\xi_M, \xi_1 - \sum_{k\neq 1}^{N} \xi_k - (N-1)\xi_2\},$$

then, since $\tilde{\xi} < 0$ by assumption, we have

$$\frac{d}{dt}(\mathcal{G}_1 + \mathcal{G}_2) \leq \frac{2\kappa}{N}\left[\tilde{\xi} - \xi_1\left(\mathcal{G}_1 + \mathcal{G}_2\right)\right](\mathcal{G}_1 + \mathcal{G}_2),$$

which is the Riccati differential inequality. Hence we obtain

$$(\mathcal{G}_1 + \mathcal{G}_2) \leq \left[\left(\frac{1}{\mathcal{G}_1^0 + \mathcal{G}_2^0} - \frac{\xi_1}{\tilde{\xi}}\right)e^{-\frac{2\kappa\tilde{\xi}}{N}t} + \frac{\xi_1}{\tilde{\xi}}\right]^{-1},$$

where the initial data satisfy

$$\mathcal{G}_1^0 + \mathcal{G}_2^0 \leq \frac{\tilde{\xi}}{\xi_1}.$$

So, we get the desired convergence result. □

Finally, we assume that all communication weights are negative: $\xi_1 < \cdots < \xi_N < 0$. Even in this case, since all the values of ξ are negative, the particles repeal each other. But if we give proper assumption, we can show the bi-polar synchronization.

Theorem 4.2.4. *Suppose that the coupling strength, communication weight and initial data satisfy*

$$\kappa > 0, \quad \xi_1 < \cdots < \xi_N < 0, \quad \tilde{\xi} < 0, \quad \mathcal{G}_1^0 + \mathcal{G}_2^0 \leq \frac{\tilde{\xi}}{\xi_1},$$

where

$$\tilde{\xi} := \max\{\xi_1 - (N-1)\xi_N, \xi_1 - \sum_{k\neq 1}^{N} \xi_k - (N-1)\xi_2\}.$$

Then, for any solution h_{ij} to (4.0.1)*, there exists a positive constant C such that*

$$\mathcal{G}_1 + \mathcal{G}_2 \leq e^{-Ct} \quad as \quad t \to \infty.$$

Proof. In this case, the process is similar to the above way. So we will look briefly. Since $\xi_1 < \cdots < \xi_N < 0$, we have

$$\begin{aligned}\frac{d}{dt}|X_l| \le \frac{2\kappa}{N}&\left(\xi_1|X_l| - \xi_N\sum_{k\neq 1}^{N}|X_k| - \xi_2\sum_{k\neq 1,l}^{N}|Y_{kl}|\right)\\ &+\frac{\kappa}{N}\left(-\xi_1|X_l| - \xi_2\sum_{k\neq 1}^{N}|X_k| - \xi_N\sum_{k\neq 1,l}^{N}|Y_{kl}|\right)|X_l|,\\ \frac{d}{dt}|Y_{lm}| \le \frac{\kappa}{N}&\left[2\left(\xi_1 - \sum_{k\neq 1}^{N}\xi_k\right) - \xi_1(|X_l| + |X_m|)\right]|Y_{lm}|\\ &-\frac{\kappa\xi_N}{N}\left(\sum_{k\neq 1,l}^{N}|Y_{lk}| + \sum_{k\neq 1,m}^{N}|Y_{km}|\right)|Y_{lm}|.\end{aligned}$$

Then, by definition of $\mathcal{G}_1, \mathcal{G}_2$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_1 &\le \frac{2\kappa}{N}\left[\xi_1 - (N-1)(\xi_N\mathcal{G}_1 + \xi_2\mathcal{G}_2)\right] + \frac{\kappa}{N}\left[-(\xi_1+\xi_2)\mathcal{G}_1 - \xi_N\mathcal{G}_2\right]\mathcal{G}_1,\\ \frac{d}{dt}\mathcal{G}_2 &\le \frac{2\kappa}{N}\left[\left(\xi_1 - \sum_{k\neq 1}^{N}\xi_k\right) - \xi_1\mathcal{G}_1 - \xi_N\mathcal{G}_2\right]\mathcal{G}_2.\end{aligned}$$

If we set

$$\tilde{\xi} := \max\{\xi_1 - (N-1)\xi_N, \xi_1 - \sum_{k\neq 1}^{N}\xi_k - (N-1)\xi_2\},$$

then, since $\tilde{\xi} < 0$ by assumption, we have

$$\frac{d}{dt}(\mathcal{G}_1 + \mathcal{G}_2) \le \frac{2\kappa}{N}\left[\tilde{\xi} - \xi_1(\mathcal{G}_1 + \mathcal{G}_2)\right](\mathcal{G}_1 + \mathcal{G}_2),$$

which is the Riccati differential inequality. Hence, we obtain

$$(\mathcal{G}_1 + \mathcal{G}_2) \le \left[\left(\frac{1}{\mathcal{G}_1^0 + \mathcal{G}_2^0} - \frac{\xi_1}{\tilde{\xi}}\right)e^{-\frac{2\kappa}{N}\tilde{\xi}t} + \frac{\xi_1}{\tilde{\xi}}\right]^{-1},$$

where the initial data satisfy

$$\mathcal{G}_1^0 + \mathcal{G}_2^0 \le \frac{\tilde{\xi}}{\xi_1}.$$

Therefore, we obtain the desired convergence result. □

Remark 4.2.1. *In this Chapter, we showed the existence of stable equilibria of the coupled S-L system. Therefore, there is no closed orbit in our assumption.*

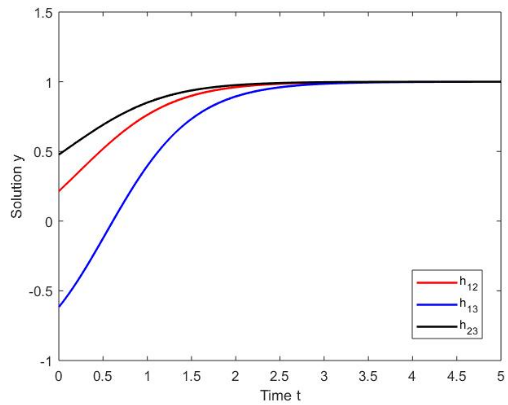
Chapter 5

Numerical simulations

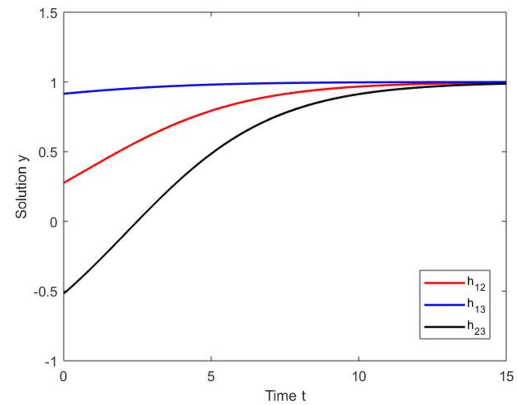
In this Chapter, we present some numerical simulations of the results proven in Chapter 4. The numerical simulations via MATLAB show that the complete synchronization and bi-polar synchronization depend on the communication weight. Note that throughout this Chapter, we restrict h_{ij} on real line and we assume that coupling strength κ is 1.

5.1 Simulation on Low-dimensional system

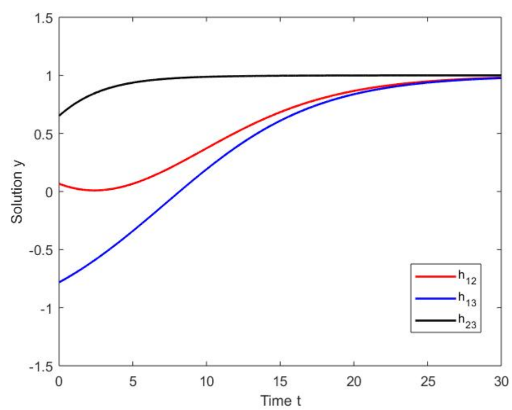
In this section, we show some numerical simulations when $N = 3$. As mentioned in the previous Chapter 4, we focus on Ξ. First, we assume that Ξ is positive.



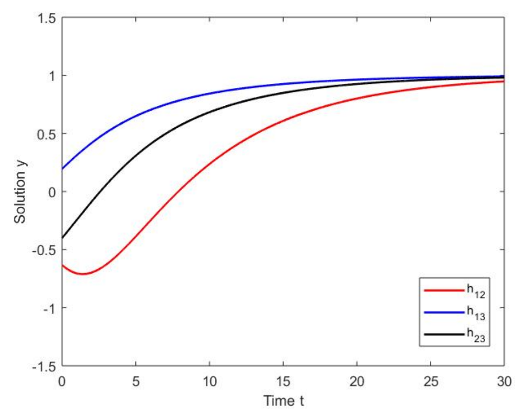
(a) $\xi = (1, 1, 1)$



(b) $\xi = (0.1, 0.2, 0.3)$



(c) $\xi = (-0.2, 0.1, 0.4)$



(d) $\xi = (-0.2, -0.1, 0.5)$

Figure 5.1: Convergence values in four different cases

Figure 5.1 simulated at $h = 0.01$. The initial data of Figure 5.1(a) are $(0.3, -0.6, 0.5)$, Figure 5.1(b) are $(0.3, 0.9, -0.5)$, Figure 5.1(c) are $(0.1, -0.8, 0.6)$ and Figure 5.1(d) are $(-0.6, 0.2, -0.4)$. As we can see from the Figure 5.1, if $\Xi > 0$, then we can show the complete synchronization.

Now, we assume that Ξ is negative. At this time, we focus on the number of negative communication weight.

1. $\xi_1 < 0 < \xi_2 < \xi_3$.

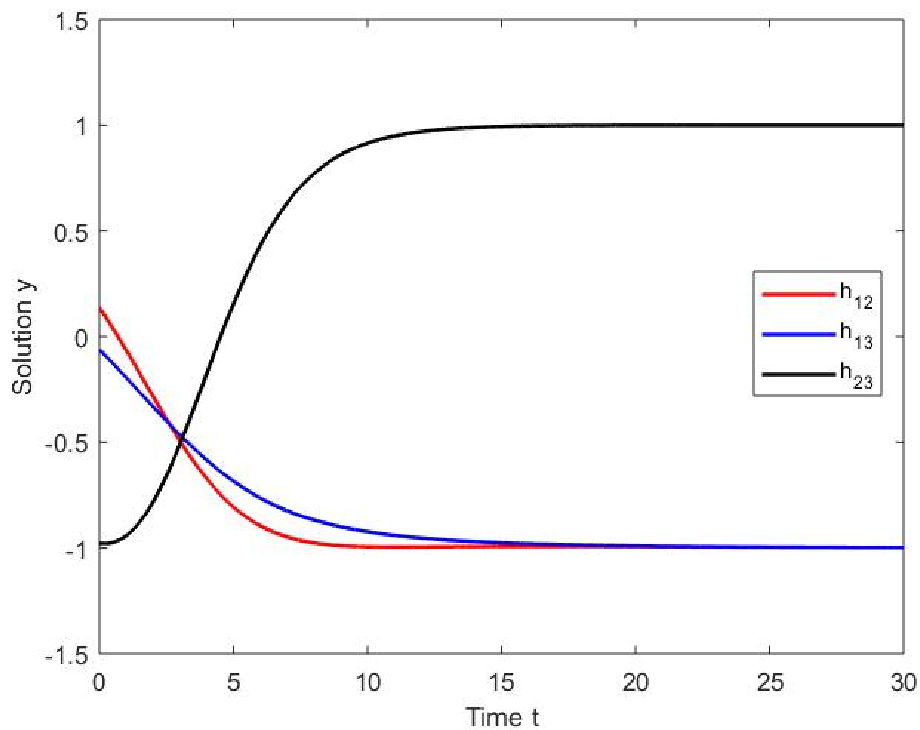
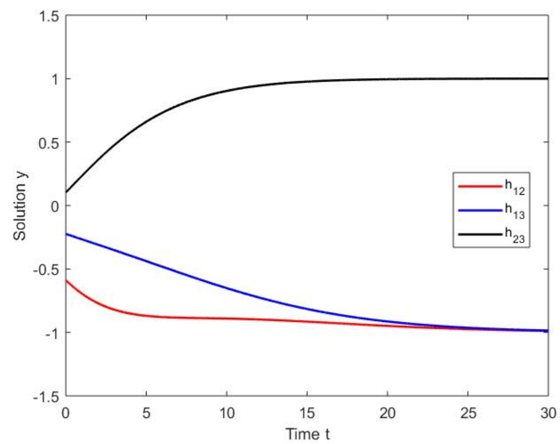


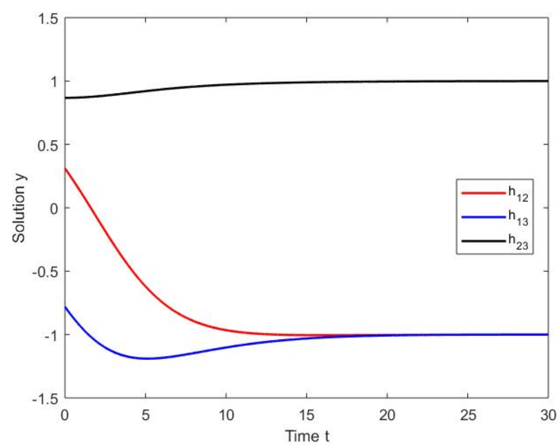
Figure 5.2: Convergence value when $\xi = (-0.5, 0.1, 0.2)$

Figure 5.2 simulated at $h = 0.01$. The initial data are $(0.2, 0.5, -0.8)$. As we can see from the Figure 5.2, we can show the bi-polar synchronization when $\xi_1 < 0 < \xi_2 < \xi_3$ and $\xi_1 < -2\xi_3$.

2. $\xi_1 < \xi_2 < 0 < \xi_3$.



(a) $\xi = (-0.4, -0.1, 0.2)$



(b) $\xi = (-0.4, -0.2, 0.1)$

Figure 5.3: Convergence value in two different cases

Figure 5.3 simulated at $h = 0.01$. The initial data of Figure 5.3(a) are $(-0.6, -0.2, 0.1)$ and Figure 5.3(b) are $(0.3, -0.8, 0.9)$. As we can see from the Figure 5.3, we can show the bi-polar synchronization whether it is $\xi_2 + \xi_3 > 0$ or $\xi_2 + \xi_3 < 0$.

3. $\xi_1 < \xi_2 < \xi_3 < 0$.

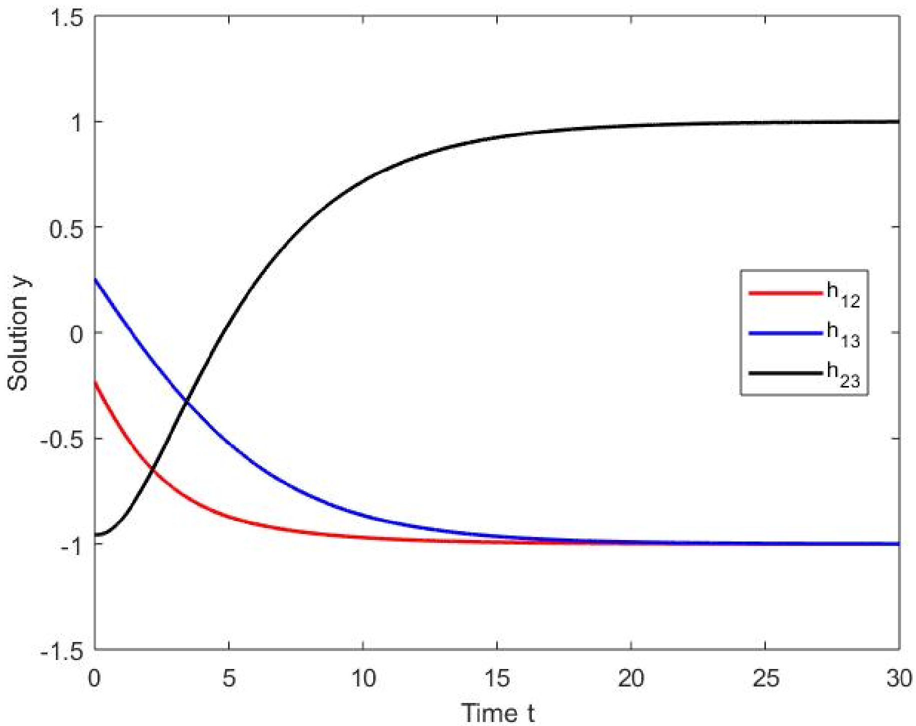
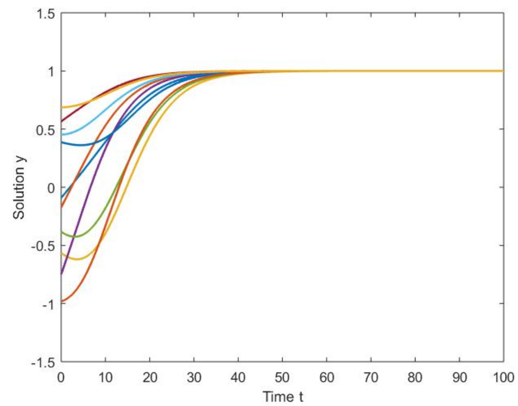


Figure 5.4: Convergence value when $\xi = (-0.7, -0.2, -0.1)$

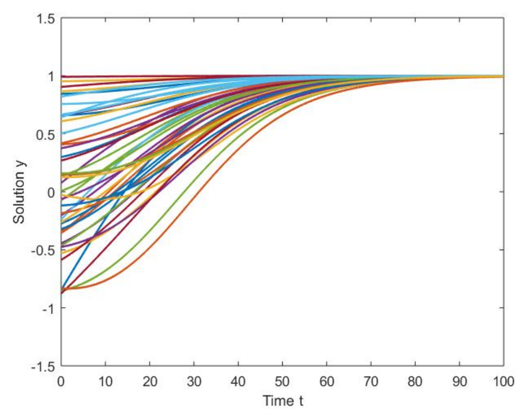
Figure 5.4 simulated at $h = 0.01$. The initial data of Figure 5.4(a) are $(-0.1, 0.2, -0.9)$. As we can see from the Figure 5.3, when $\xi_1 < 2(\xi_2 + \xi_3) < 0$, we can show the bi-polar synchronization.

5.2 Simulation on Large-dimensional system

In this section, we show some numerical simulations for general N. We focus on Ξ as in the low-dimension cases. So first, we assume that Ξ is positive.



(a) $N = 5$



(b) $N = 10$

Figure 5.5: Convergence values in two different cases

Figure 5.5 simulated at $h = 0.05$. Ξ in the Figure 5.5(a) is 2.1 and Figure 5.5(b) is 3.8. As we can see from the Figure 5.5, for each N, we can show the complete synchronization.

Now, we assume that Ξ is negative. In this case, we focus on the number of negative communication weight.

1. $\xi_1 < 0 < \xi_2 < \cdots < \xi_N$.

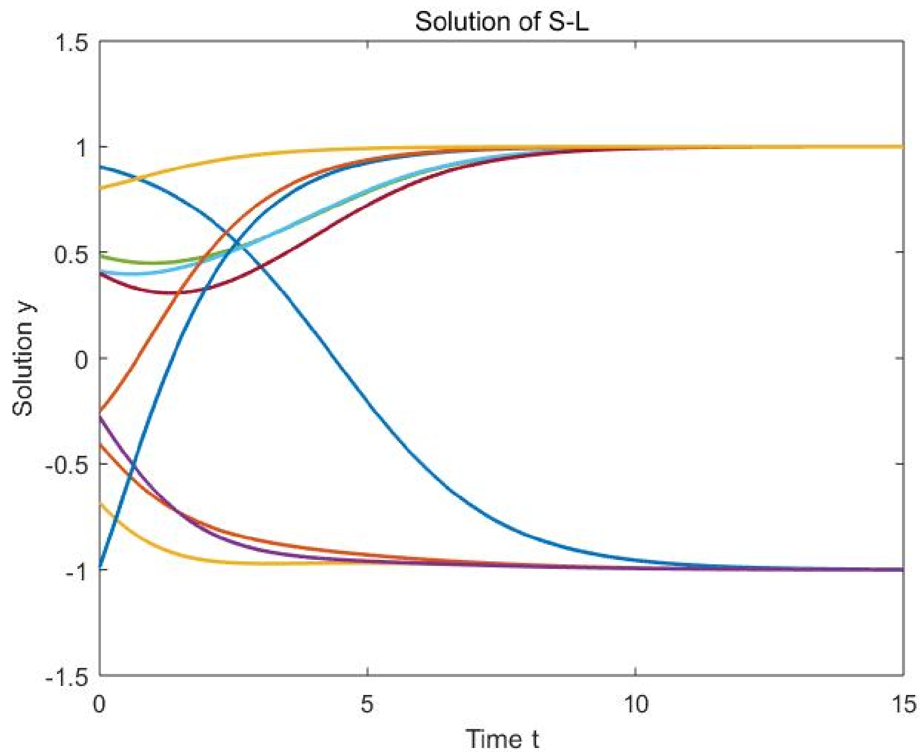


Figure 5.6: Convergence value when $N = 5$

Figure 5.6 simulated at $h = 0.01$ and Ξ is -0.8. As we can see from the Figure 5.6, if Ξ is negative and $\xi_1 + 4\xi_5 < 0$, particles are moved by ξ_1. So h_{1l} converges to -1 and h_{lm} converges to 1 for $2 \leq l, m \leq 5$. Hence we can show the bi-polar synchronization.

2. $\xi_1 < \cdots < \xi_n < 0 < \xi_{n+1} < \cdots < \xi_N$ for some $n \neq 1, N$.

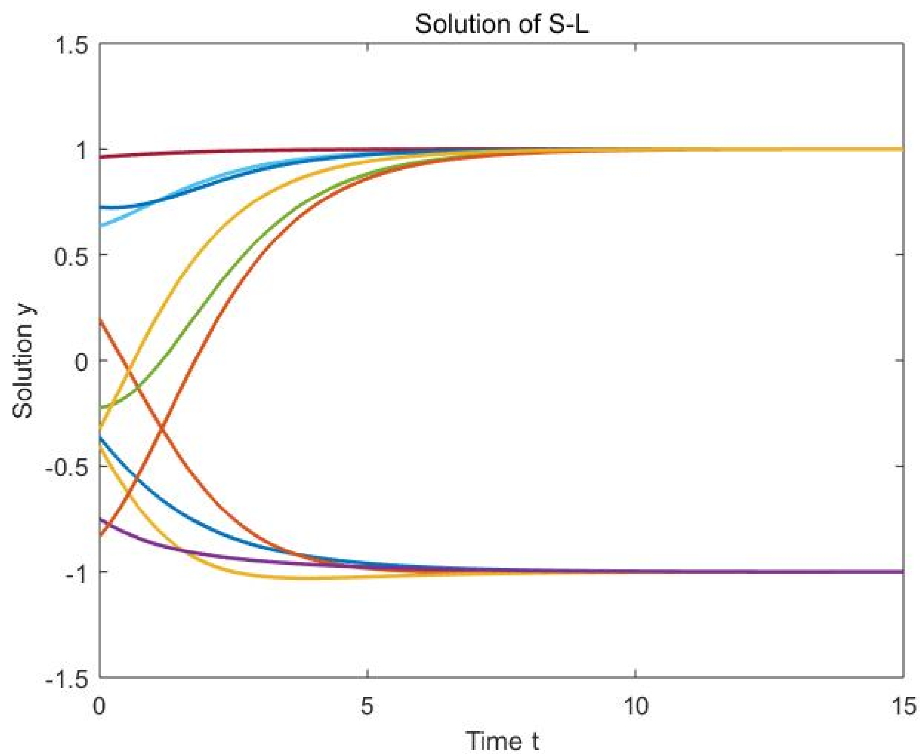


Figure 5.7: Convergence value when $N = 5$

Figure 5.7 simulated at $h = 0.01$. Ξ is -1.7 and $n = 3$. As we can see from the Figure 5.7, if $\xi_1 < \xi_2 < \xi_3 < 0 < \xi_4 < \xi_5$ and $\xi_1 + 4\xi_M < 0$, then we can show the bi-polar synchronization.

3. $\xi_1 < \cdots < \xi_N < 0$.

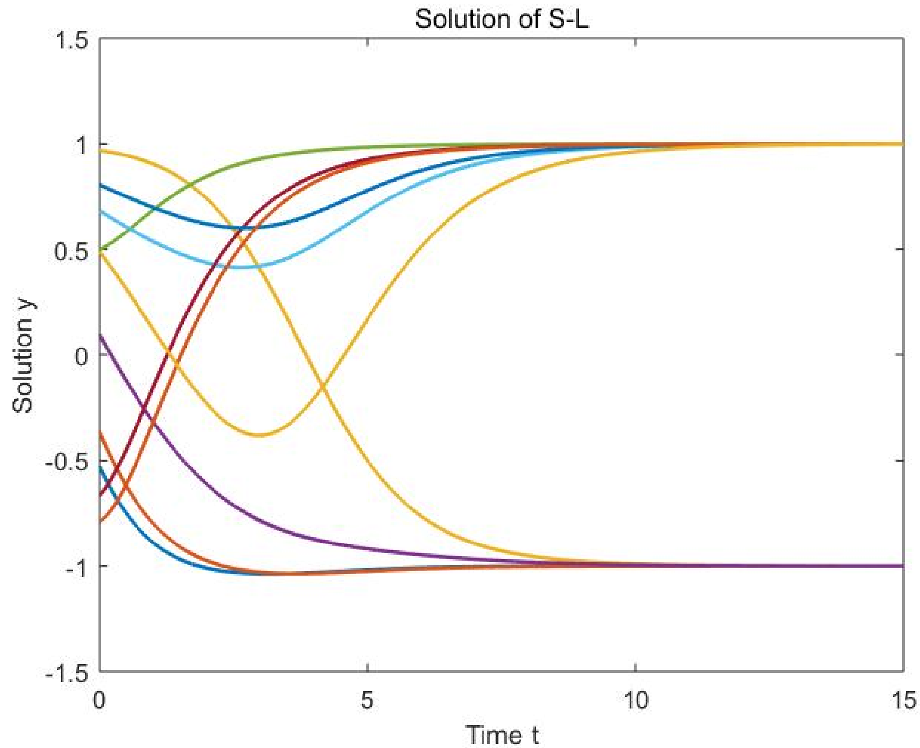


Figure 5.8: Convergence value when $N = 5$

Figure 5.8 simulated at $h = 0.01$ and Ξ is -3.7. Indeed, particles repeal each others in this case. But if $\xi_1 - 5\xi_2 - \xi_3 - \xi_4 - \xi_5 < 0$, then the network dominated by ξ_1. So we can see the bi-polar synchronization.

Chapter 6

Conclusion

In this thesis, we present the existence of equilibria of coupled Schrödinger-Lohe system for finite-dimension. First, we study the steady state of the Lotka-Volterra system and relation between the Schördinger-Lohe model to the Kuramoto model. Then we briefly review the previous result on the wave function synchronization when Schrödinger-Lohe system has all-to-all cooperative couplings. We introduce the pairwise correlation function to reduce the Schrödinger-Lohe system which is related to Lotka-Volterra system. So far, Schrödinger oscillators are studied under the all-to-all cooperative interaction or pure interaction case. In this thesis, we figure out all the stable equilibria of coupled Schrödinger-Lohe system when the communication weight attain both positive and negative value. We also provide the numerical validation supporting our analytical results. However, we need too much constraint on the communication weight to show the synchronization. In the numerical simulation, we found that these assumptions are too strong for the synchronization and maybe the same analytical results still hold without those assumptions. We left this for future works.

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